Faster submodular optimization for several matroids

Using dynamic data structures to speed up optimization problems

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Submodular functions

For a set $\mathcal{N}$, a set function $f : 2^{\mathcal{N}}$ is **submodular** if for any sets S and T:

$$f(S) + f(T) \geq f(S + T) + f(S - T)$$

We only consider monotone submodular functions.

$$f(S) \leq f(T) \quad \text{for any sets } S \subseteq T$$

We assume we have an oracle that computes f.

Matroid

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2. Exchange property:

$$\forall S, T \in \mathcal{I} \text{ and } |S| < |T| \Rightarrow \exists e \in T \setminus S \text{ such that } S + e \in \mathcal{I}$$

Submodular maximization with matroid constraints

Goal: Given and a matroid $\mathcal{M} = (\mathcal{N}, \mathcal{I})$ and a submodular f over $\mathcal{N}$, find:

$$S^* = \arg\max_{S \subseteq \mathcal{I}} f(S)$$

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- Submodular optimization is a fundamental problem in combinatorial optimization, information retrieval, and machine learning.
- Many applications involve combinatorial constraints on subsets, matroids are very general ones that have been well studied.

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3. **Laminar** - A tree representing the laminar family
   Application: Capacity constraints - YouTube recommendations [WRB+18]

Prior Results and New Results

n is the size of the ground set, m is the size of the matroid representation,

r is the rank of the matroid.

* Assumes access to an independence oracle

Constraint	Runtime	Paper	New
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Takeaway message

Efficient dynamic data structures

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$\downarrow\downarrow\downarrow$ $\uparrow\uparrow\uparrow$

Solving optimization problems faster

Our results

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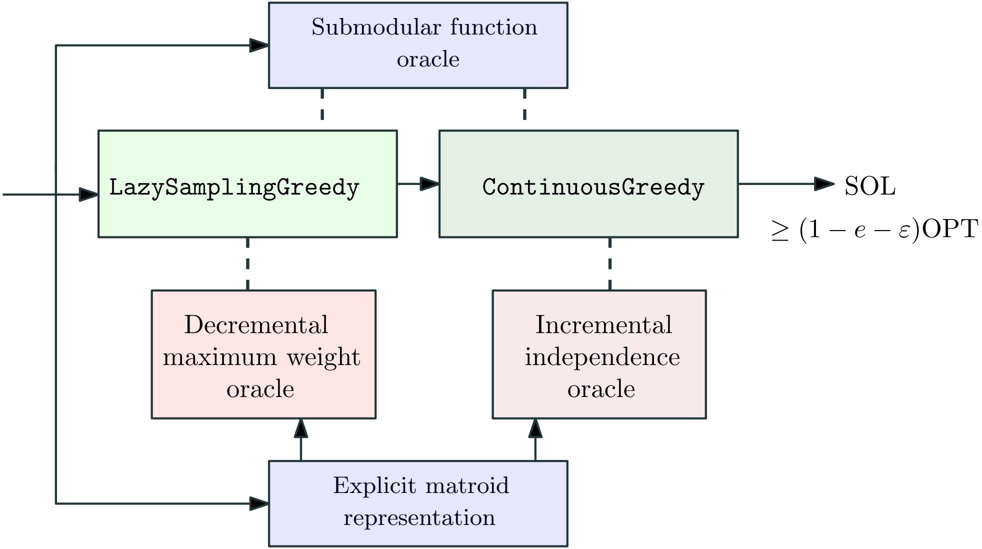
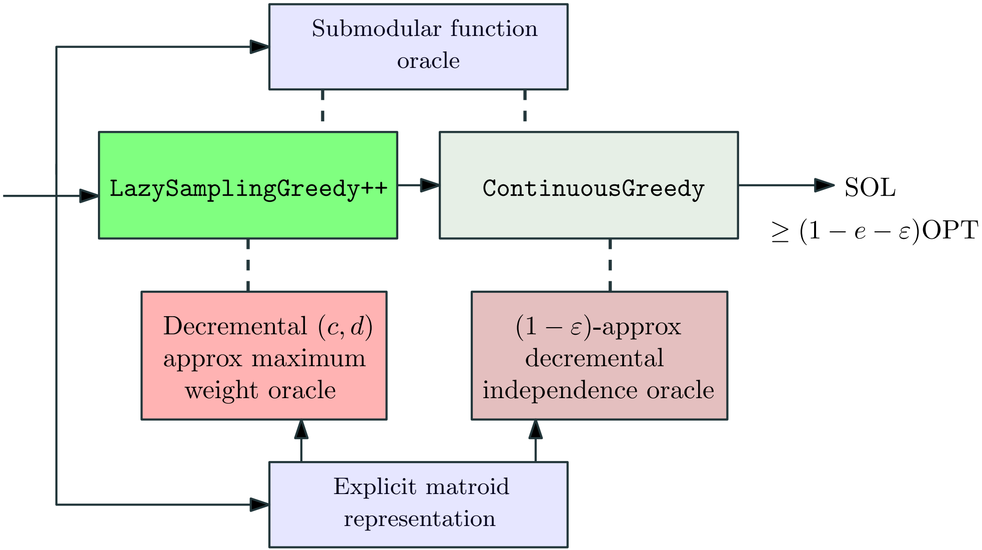
3. A data structure for maintaining a $(1-\varepsilon)$–approximate maximum weight matchings in a vertex weighted graph (transversal matroid) with weight decrement operations.

Framework of Ene & Nguyen '19



New Framework



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- test adding an element e, outputs if $B + e$ is independent,
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Laminar Matroids

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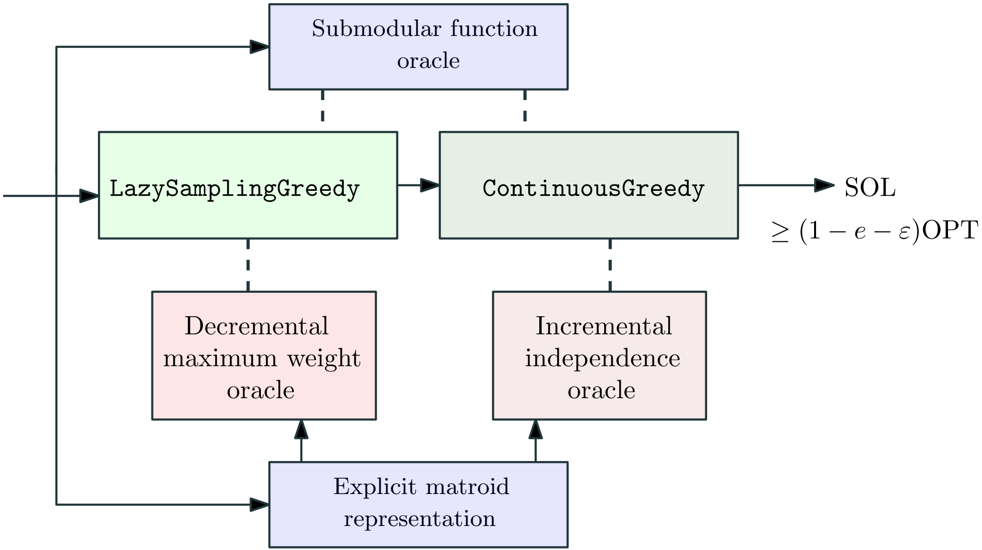

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Goal: the maximum weight set of leaves that respect the capacity constraints.

Laminar

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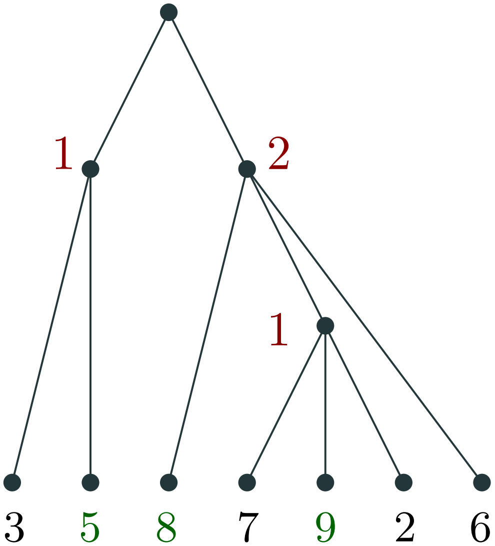
Theorem. There exists a data structure that supports the insertion and deletion of leaves of arbitrary weight that maintains the maximum feasible set of leaves with $O(\log n)$ update time.

Laminar example



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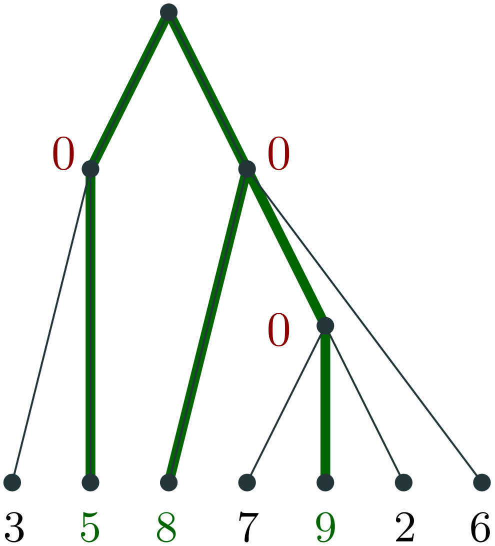

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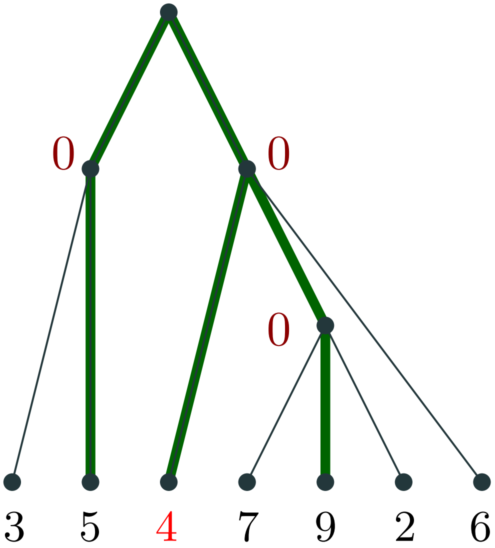

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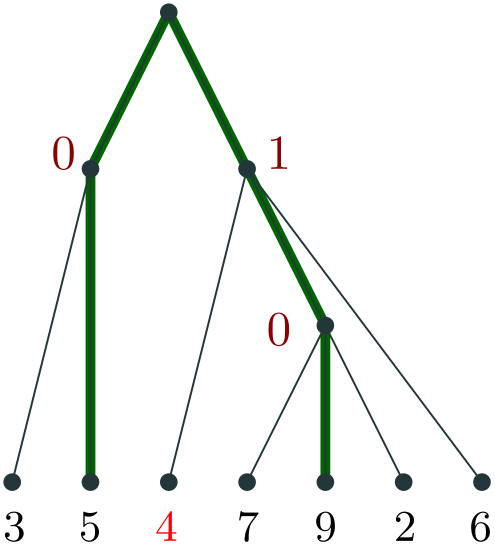

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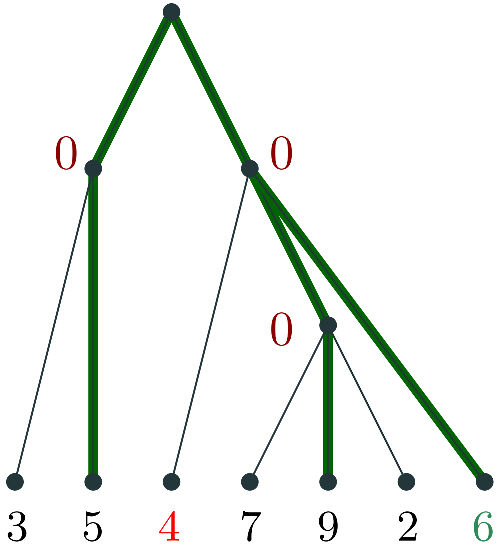

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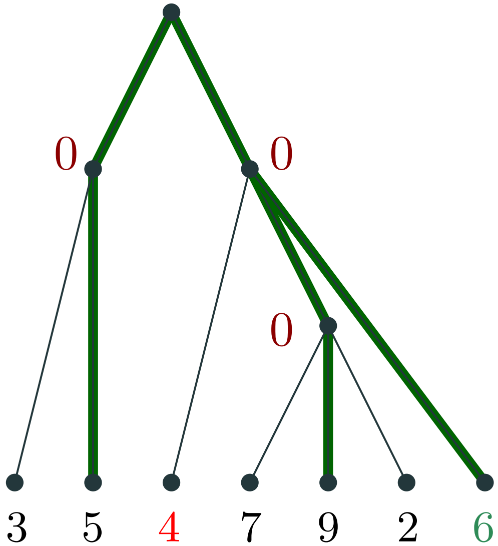

$O(n)$ time per update -

Not too hard

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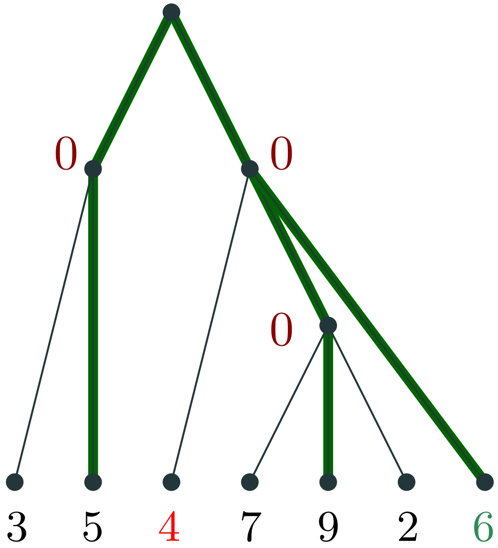

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$O(\log^2 n)$ time per update - Heavy-light decomposition

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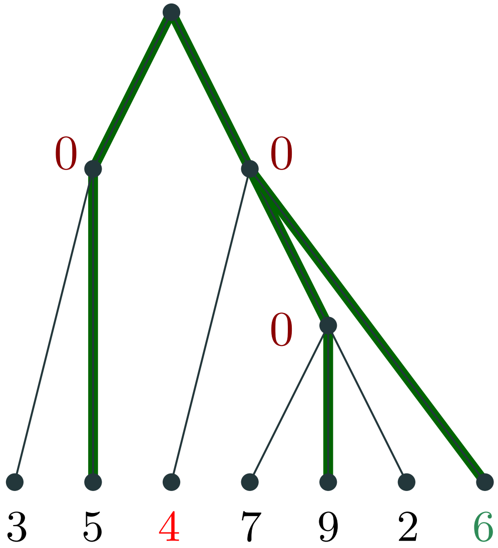

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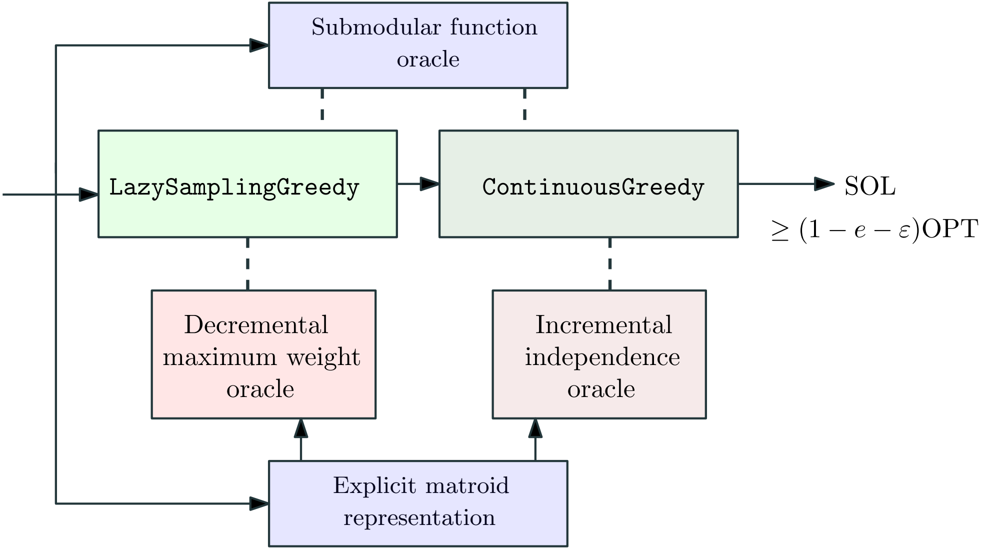
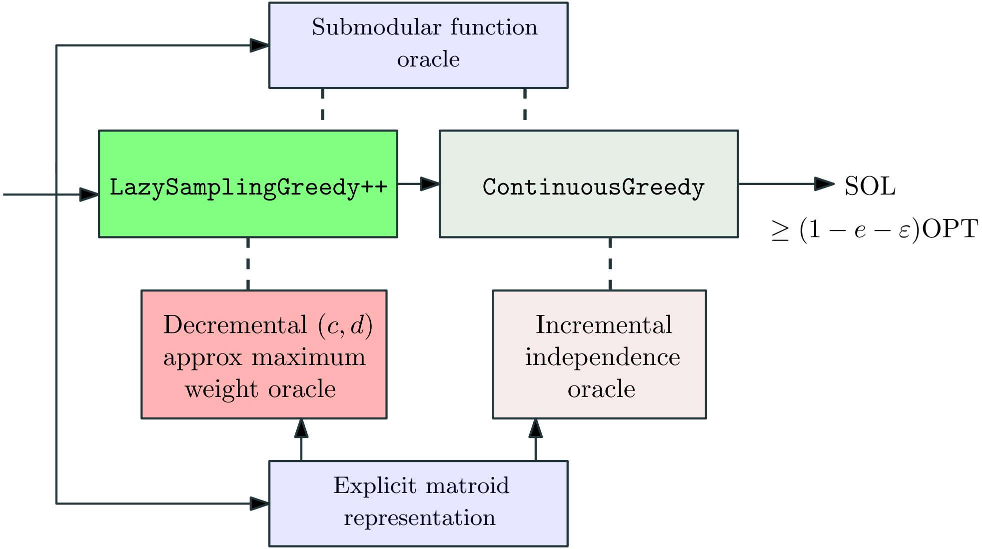
$O(\log n)$ time per update - Top tree framework

Graphic Matroids

Framework of Ene & Nguyen ’19



New Framework



Improved Framework

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New framework

1. Use algorithm `LazySamplingGreedy++` to sample from constant approximate independent set of constant size $O_\varepsilon(\log n)$ times as a preconditioning step.

Data structure differences

Old framework

Decremental maximum weight oracle

Given weights w over elements of $\mathcal{N}$ maintain the maximum independent set B and:

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New framework

Decremental (c, d)-approximate maximum weight oracle
Given weights w over elements of $\mathcal{N}$, constants $c < 1$ and $d < 1$, maintain an independent set B that is:

- a c-approximate of the maximum independent set,
- has at least dr elements,
- supports decrementing the weight of elements of $\mathcal{N}$.
- supports "freezing" an element (required to be in B)

Graphic approximate maximum weight oracle

Dynamic $(1/2, 1/2)$-approximate maximum weight oracle

Given weights w over edges E of a graph $G = (V, E)$ maintain a maximum spanning tree B that is:

- a $(1/2)$-approximate of the maximum spanning tree,
- has at least $|V|/2$ edges,
- supports decrementing weight of an edge.
- supports contracting an edge

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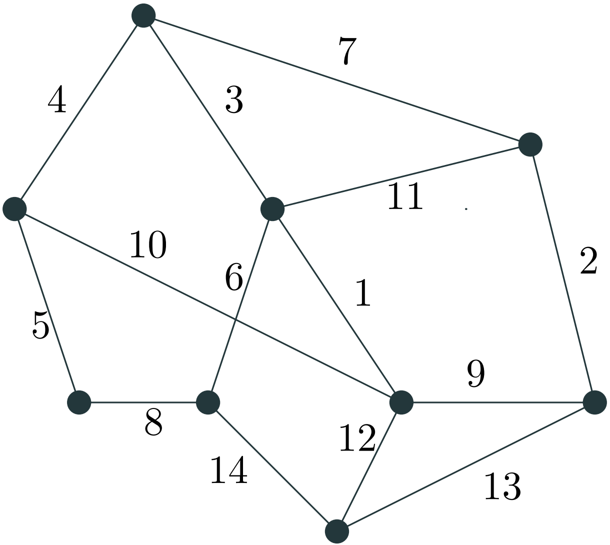
Simple data structure with $O(\log n)$ update time

- Maintain for every vertex a sorted list of incident edges,
- $B =$ union of maximum weight incident edge to each vertex
- On decrement, fix sorted list at endpoints (heap).
- On contract, merge lists (heaps)

Graphic matroid example



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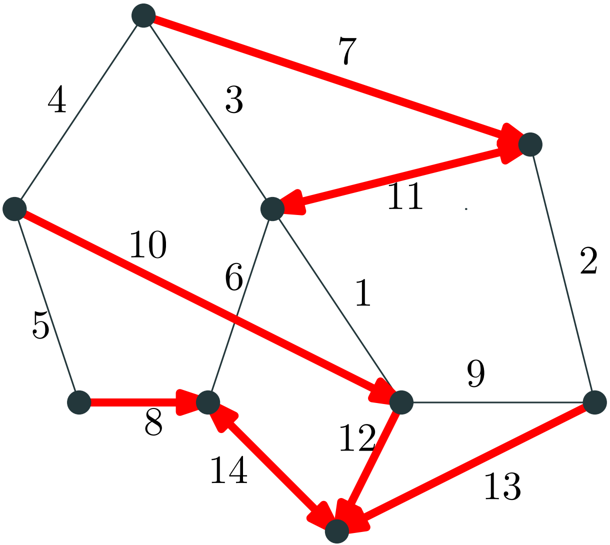

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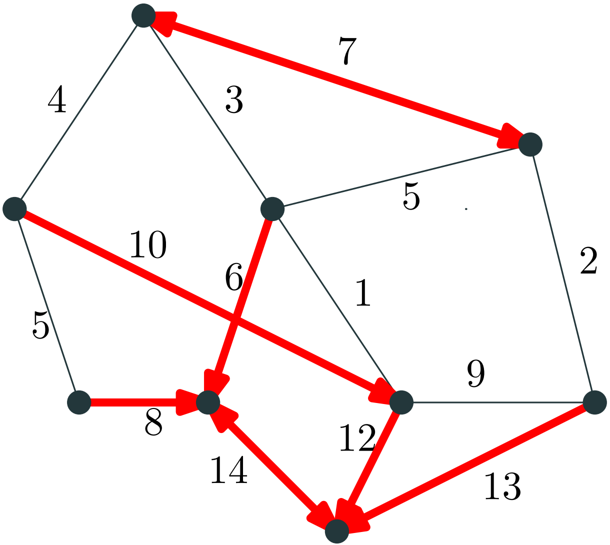
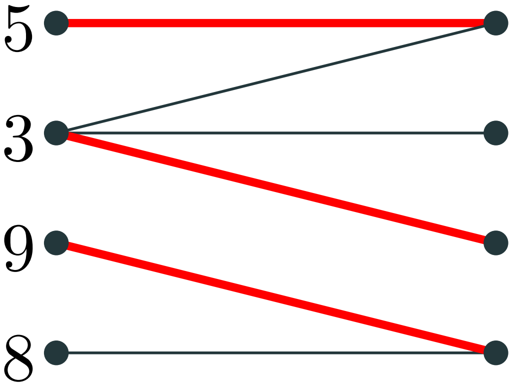


Transversal matroids

Transversal matroids = Matchable U of a bipartite graph $G = (U \cup V, E)$

Let $m = |E|$.



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Dynamic maximum weight oracle:

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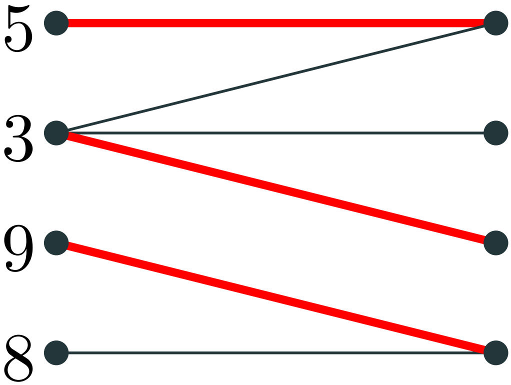

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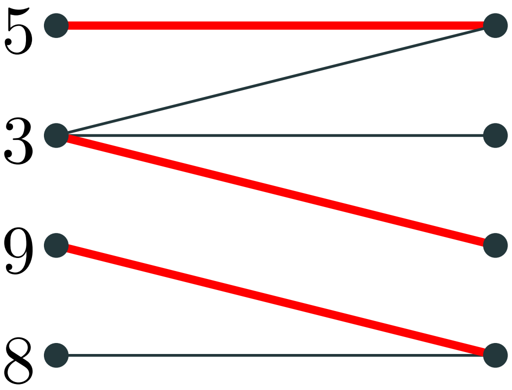

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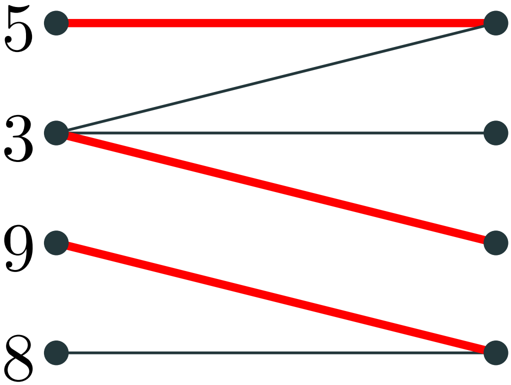
Fast algorithm would imply fast matching.
Both problems have *conditional lower bounds* of $\Omega(m^{3/2-\varepsilon})$ update time.



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New framework

$(1 - \varepsilon)$-**approximate decremental independent set data structure**

Maintains a set B of size $(1 - \varepsilon)r$ such that we can:

- delete one element from $\mathcal{N}$.

Transversal dynamic approximate maximum weight oracle

Dynamic $(1-\varepsilon, 1/2)$-approximate maximum weight oracle

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High level idea:

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High level idea:

Use variation of multiplicative auction algorithm of Z. and Henzinger (2023) to support vertex weight decrements with $O(\log n)$ update time, and ensure maximality.

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High level idea:

Use vertex decremental algorithm of Bosek, Leniowski, Sankowski, and Zych (2014)

Summary

1. Improved framework for fast submodular maximization with matroid constraints and a reduction to dynamic matroid independent set problems
2. A data structure for laminar matroids undergoing insertion and deletion updates with $O(\log n)$ update time.
3. A data structure for maintaining a $(1 - \varepsilon)$–approximate maximum weight matchings in a vertex weighted graph (transversal matroid) with weight decrement operations.

New Framework