Optimal Algorithm for Higher Order Voronoi Diagrams in 2D

The usefulness of nondeterminism

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Voronoi Diagrams

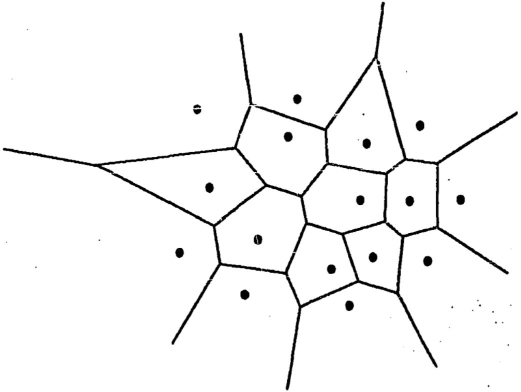


Figure 6.19: The Voronoi Diagram.

[Shamos 1978]

Order-k Voronoi Diagrams

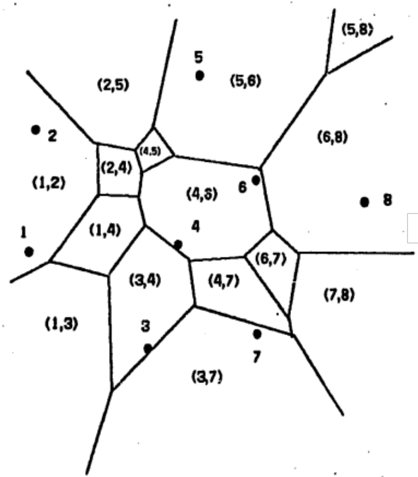


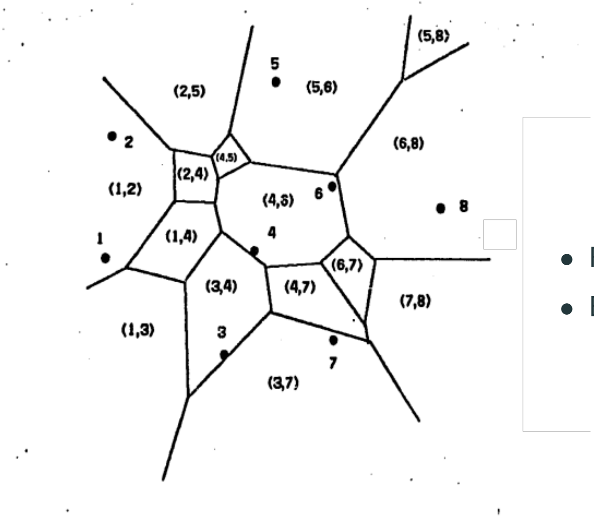
Figure 6.33: A Voronoi Diagram of Order Two.

[Shamos 1978]

Basic facts

- Fundamental (textbook) problem

Order-k Voronoi Diagrams



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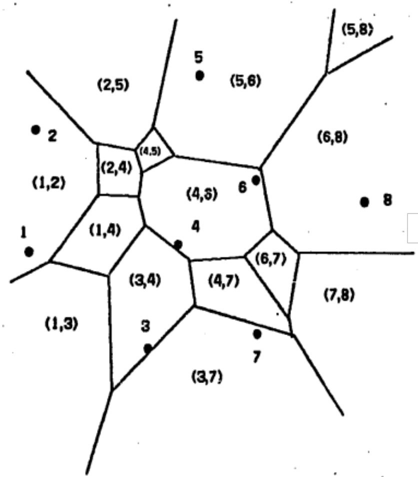

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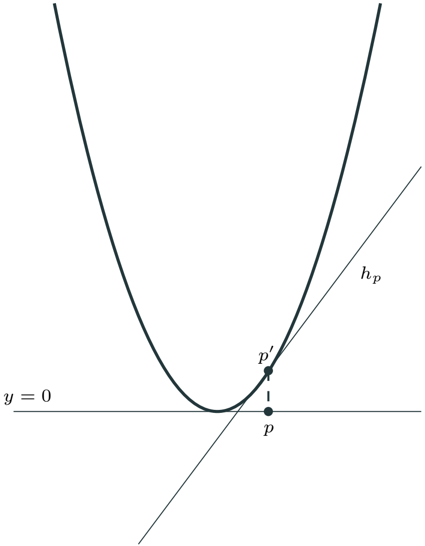
Basic facts

- Fundamental (textbook) problem
- Each region convex polygon
- Size is $\Theta(nk)$

[Shamos 1978]

Lifting and k-levels

$$p := (p_x, p_y)$$
$$p' := (p_x, p_y, p_x^2 + p_y^2)$$



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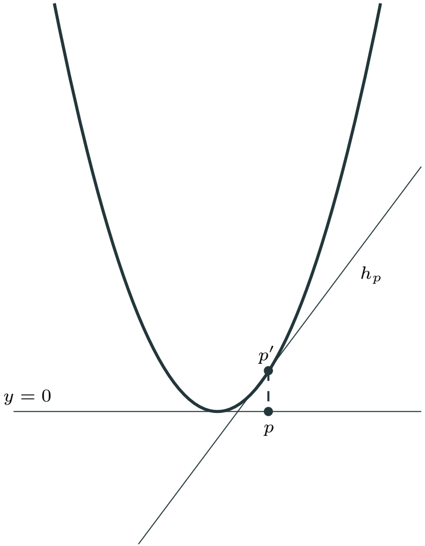

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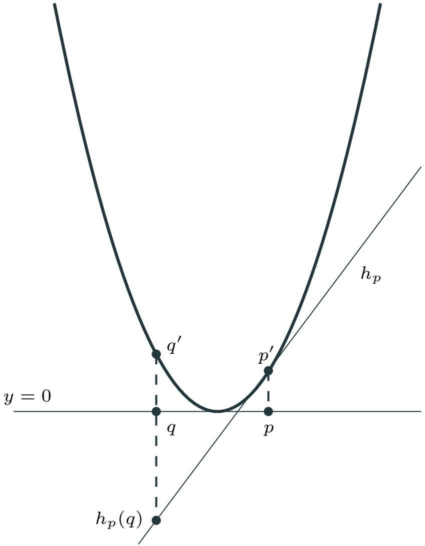

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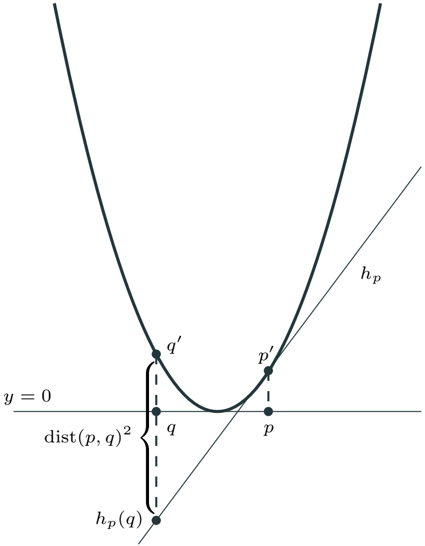

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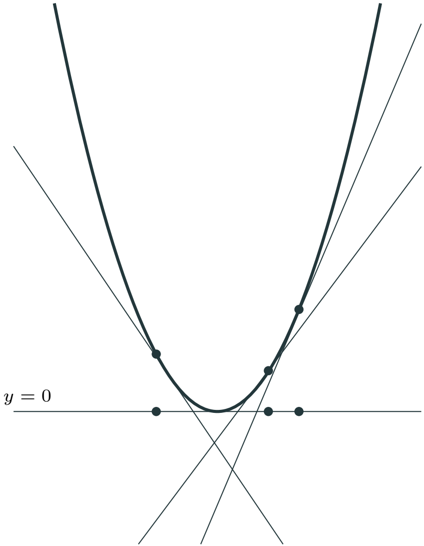

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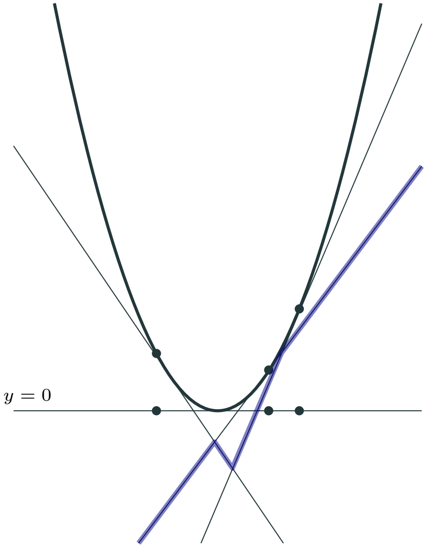

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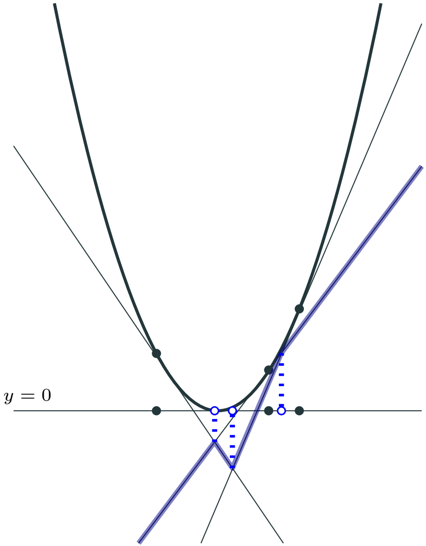

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Lifting and k-levels

$$\boxed{\text{Order } k\text{-Voronoi} = k\text{-level of 3D planes}}$$

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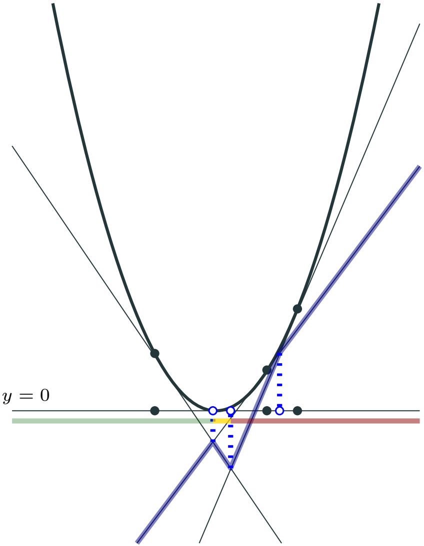

Constructing the $\{1, \ldots, k\}$-level

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$O(n^3)$	Edelsbrunner, O'Rourke, and Seidel '83

Algorithms for constructing Order-k Voronoi diagrams

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This paper:

$\boxed{O(nk + n\log n) \text{ rand.}}$ Optimal!

Self reductions

[Ramos '99] Reduces in $O(n^2)$ time rand. to $O(n^2/\log^2 n)$ problems of size $\log n$:

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Goal: Find an $O(n^2)$ time algorithm!

Another consequence of Ramos' divide and conquer

Repeat Ramos' divide and conquer until problems are of size $b = \log\log\log n$.

$$\begin{aligned} T(n) &\leq O(n^2/(\log n)^2) \cdot T(\log n) + O(n^2) \\ &\leq O(n^2/(\log\log n)^2) \cdot T(\log\log n) + O(n^2) \\ &\leq O(n^2/(\log\log\log n)^2) \cdot T(\log\log\log n) + O(n^2) \end{aligned}$$

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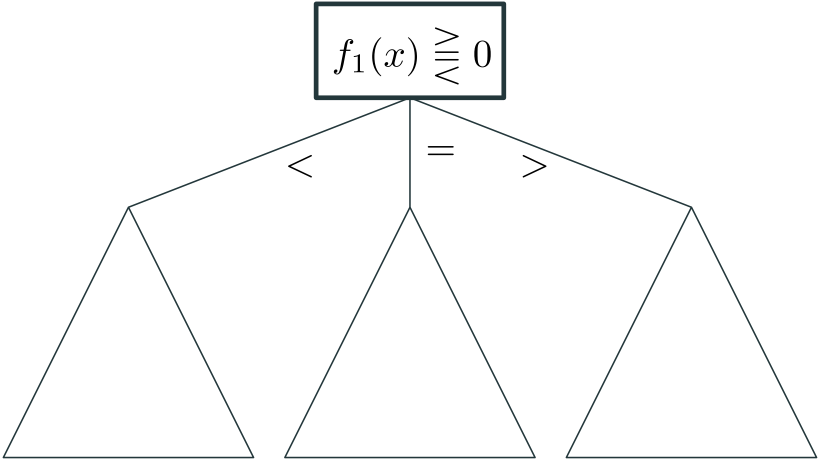
New goal: Find a quadratic depth decision tree!

Aside: On (algebraic) decision trees



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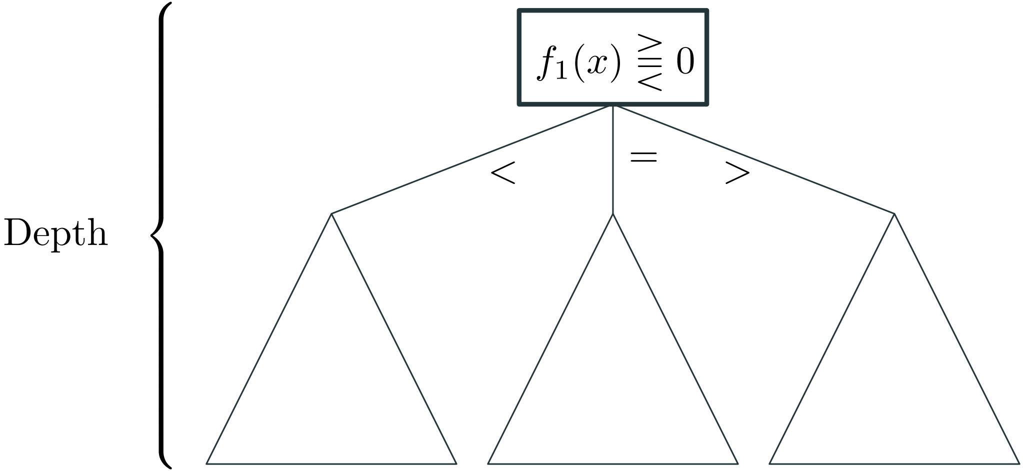

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   (Idea: "Guess" entire k-level!)
2. Solve verification problem in $O(n^2)$ time.
   (Idea: "standard" alg. w/ planar separators, recursion, dynamic 3d conv. hulls)

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(Idea 1: Use similar ideas in removing $2^{O(\log^* n)}$ factors for Hopcroft's problem)

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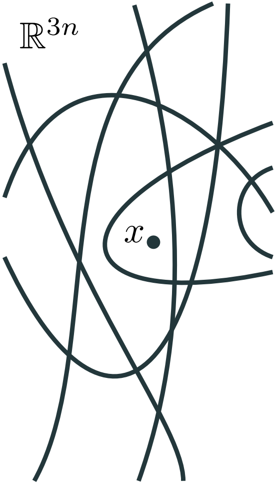

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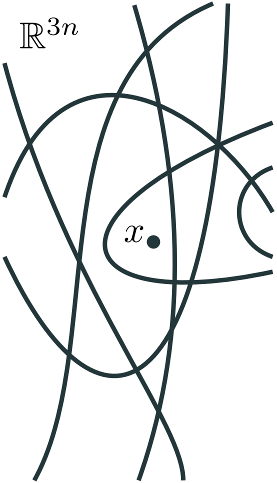
- Each comparison is a high dimensional algebraic surface.
- By Milnor-Thom, there are $(3n)^{O(3n)} = n^{O(n)}$ different cells in arrangement.



Reduction to verification problem

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Use just one round to get problems of size $b = O(\log n)$.

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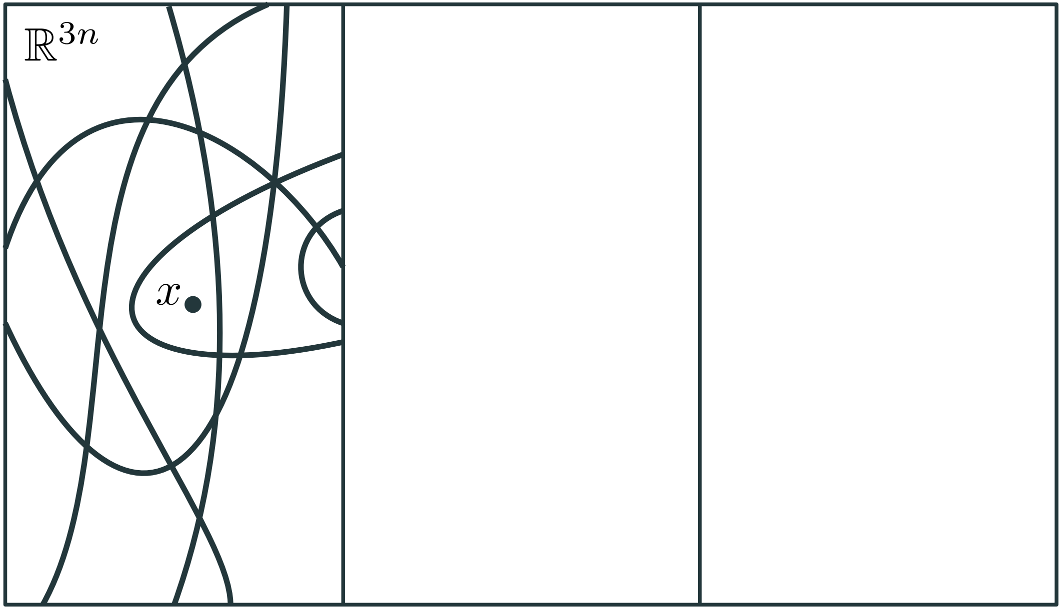
$\mathbb{R}^{3n}$
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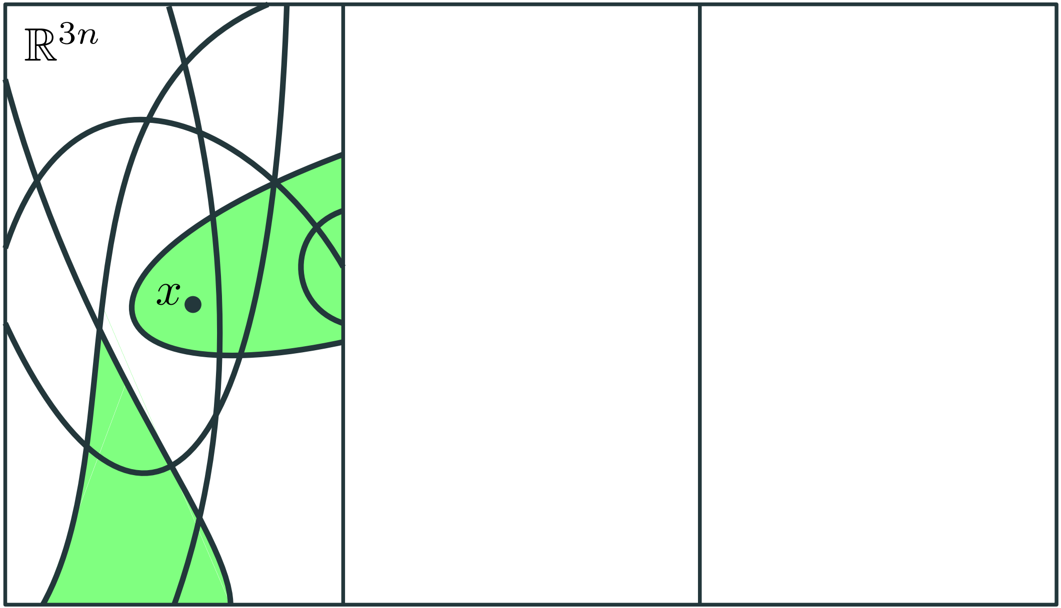
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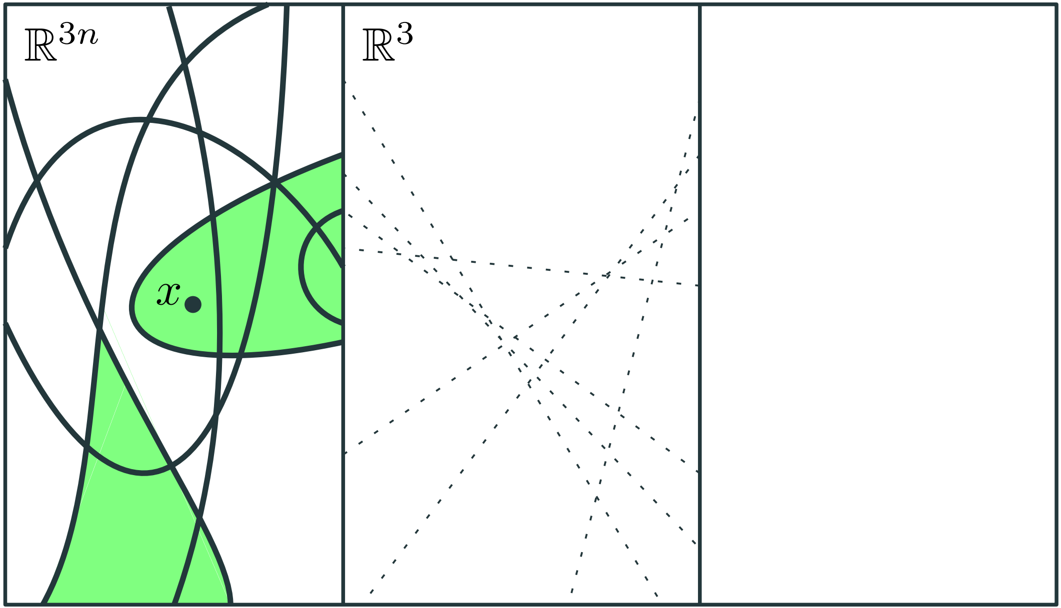

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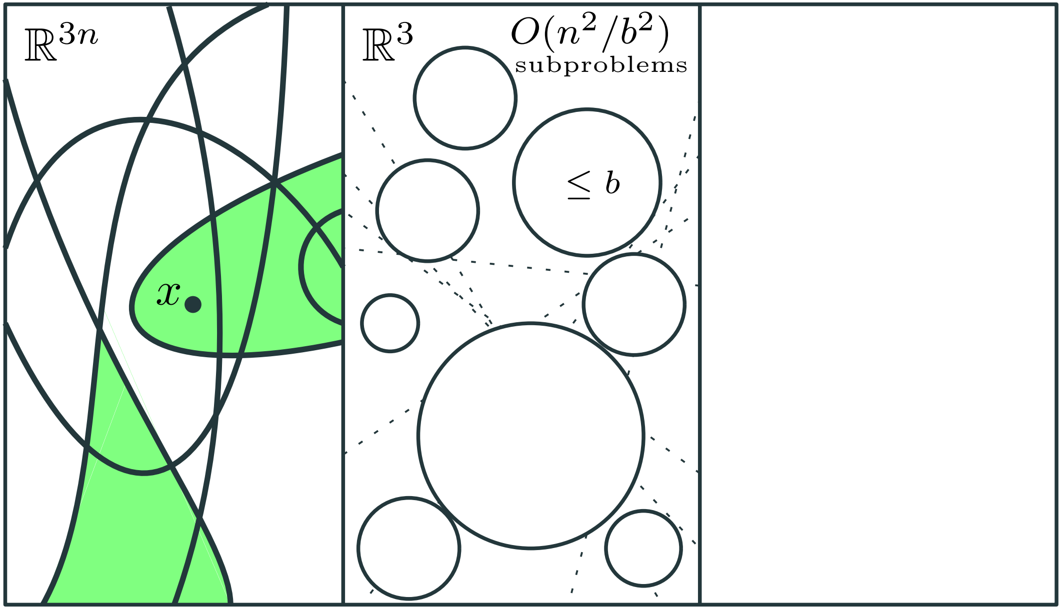
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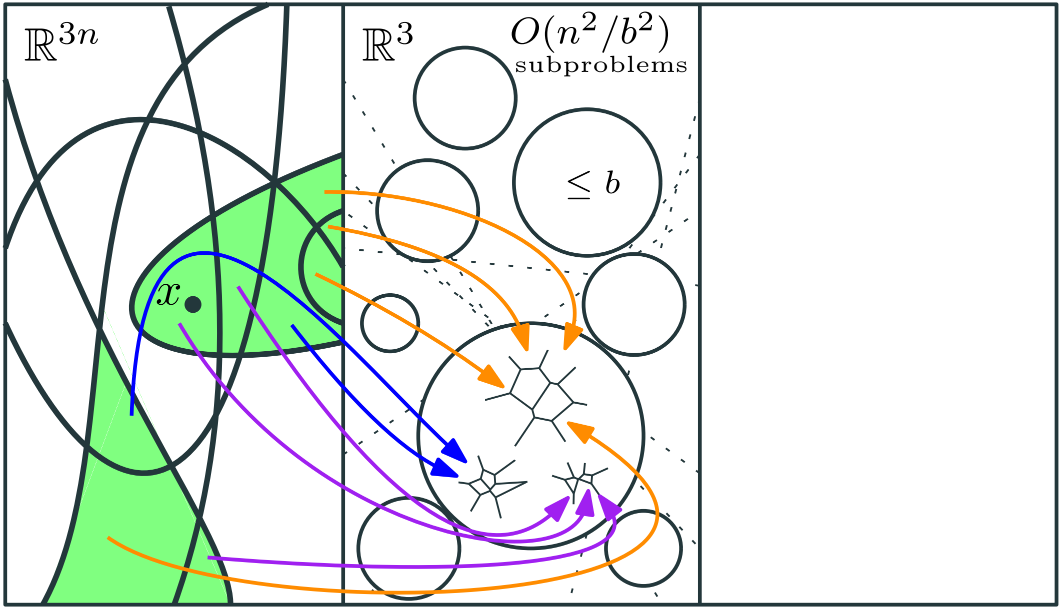
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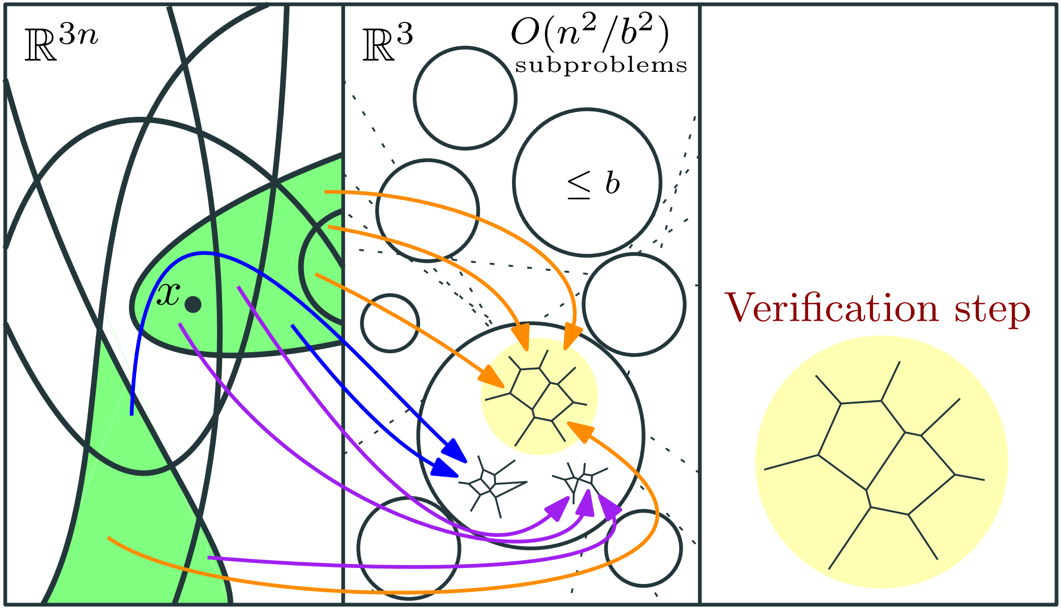

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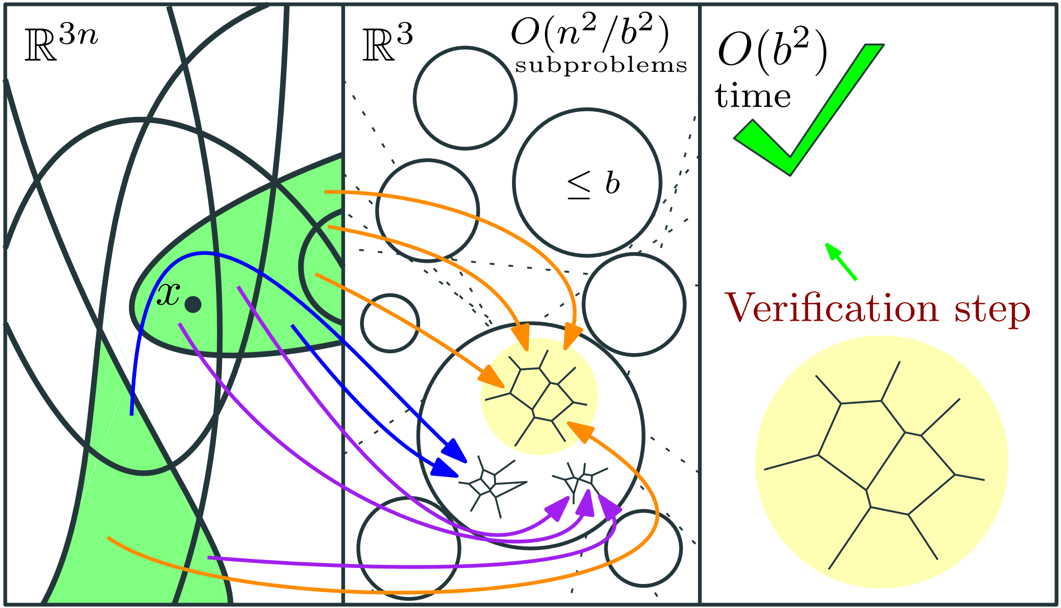
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time
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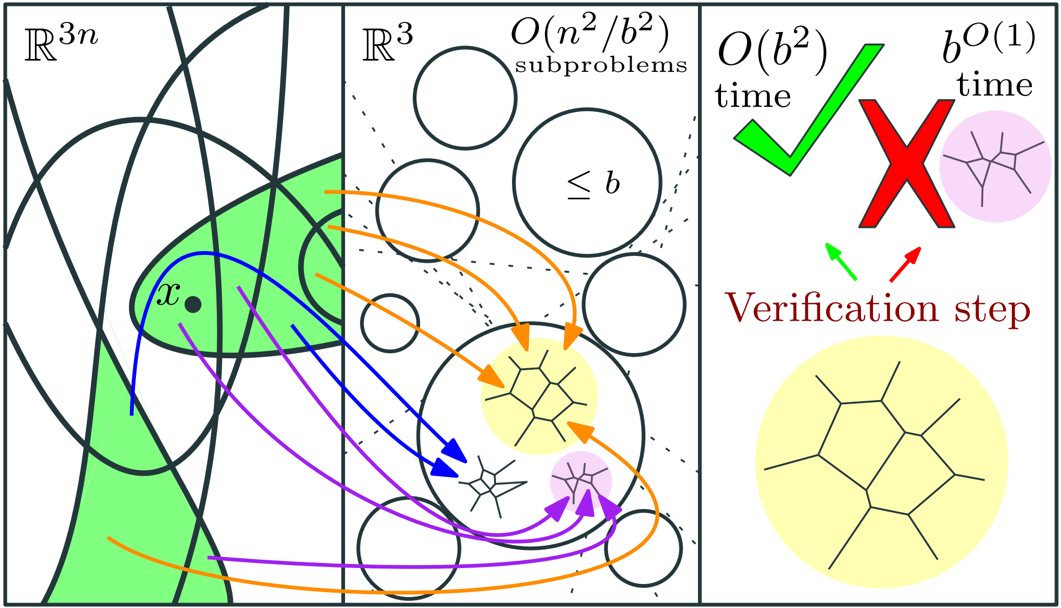

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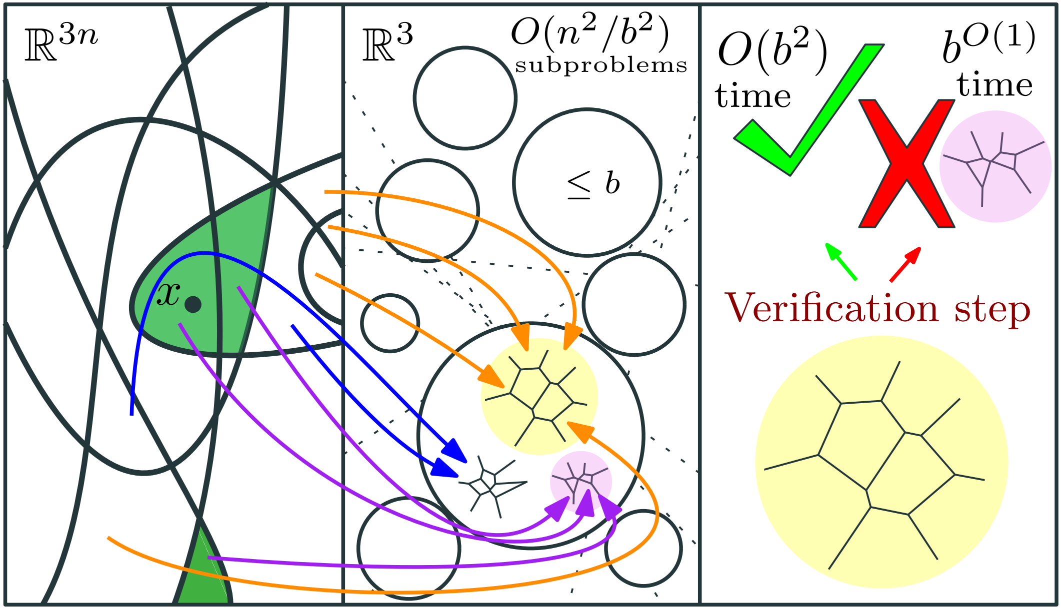
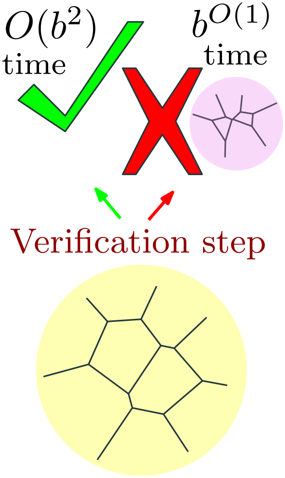
time
$b^{O(1)}$
time
Verification step


$\mathbb{R}^{3n}$
x
$\mathbb{R}^3$
$O(n^2/b^2)$
subproblems
$\leq b$
$O(b^2)$
time
$b^{O(1)}$
time
Verification step

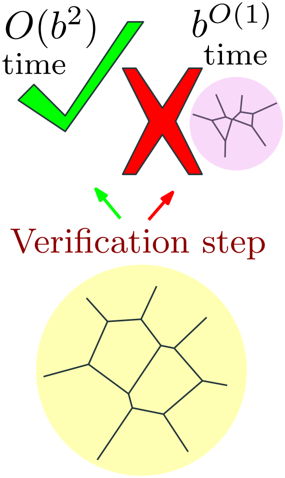
Overall runtime

- If we guessed wrong, the answer might be the second most popular option.



Overall runtime

- If we guessed wrong, the answer might be the second most popular option.
- Number of active cells decreases by factor of $1/2$.



Overall runtime

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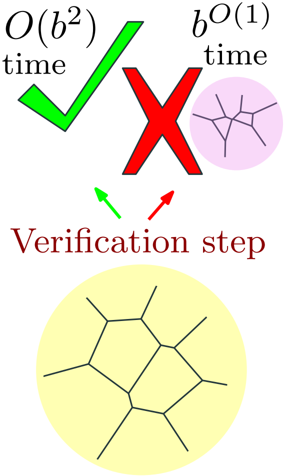
$$T(n) \leq O(n^2/b^2) \cdot O(b^2) + b^{O(1)} \cdot \log\left(n^{O(n)}\right)$$



Overall runtime

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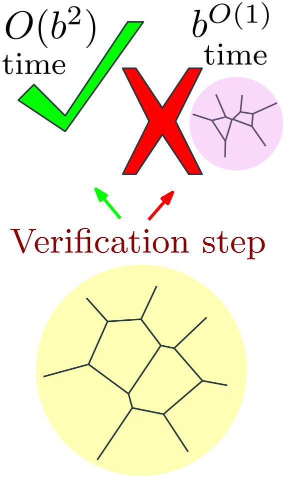
$$
\begin{aligned}
T(n) &\leq O(n^2/b^2) \cdot O(b^2) + b^{O(1)} \cdot \log\left(n^{O(n)}\right) \\
&\leq O\left(n^2 + b^{O(1)} n \log n\right)
\end{aligned}
$$



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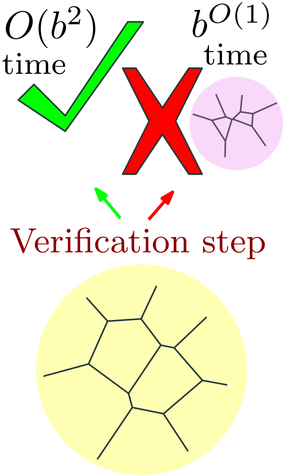
&\leq O(n^2)
\end{aligned}
$$



Open Problems

Conclusion

Open Problems

- Simpler algorithm?

Open Problems

- Simpler algorithm? (Decision trees impractical)

Open Problems

- Simpler algorithm? (Decision trees impractical)
- Optimal deterministic algorithm?

Open Problems

- Simpler algorithm? (Decision trees impractical)
- Optimal deterministic algorithm? (Ramos' randomized self-reduction)

Open Problems

- Simpler algorithm? (Decision trees impractical)
- Optimal deterministic algorithm? (Ramos' randomized self-reduction)
- Extensions of decision trees and non-determinism?

Open Problems

- Simpler algorithm? (Decision trees impractical)
- Optimal deterministic algorithm? (Ramos' randomized self-reduction)
- Extensions of decision trees and non-determinism? (Need self-reduction)

Conclusion

Open Problems

- Simpler algorithm? (Decision trees impractical)
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  - Hopcroft's problem in $o(n^{4/3})$ time?

Open Problems

- Simpler algorithm? (Decision trees impractical)
- Optimal deterministic algorithm? (Ramos' randomized self-reduction)
- Extensions of decision trees and non-determinism? (Need self-reduction)
    - Hopcroft's problem in $o(n^{4/3})$ time?
    - Detecting collinear triples in $\mathbb{R}^2$ in $o(n^2)$ time?

Conclusion

Open Problems

- Simpler algorithm? (Decision trees impractical)
- Optimal deterministic algorithm? (Ramos' randomized self-reduction)
- Extensions of decision trees and non-determinism? (Need self-reduction)
  - Hopcroft's problem in $o(n^{4/3})$ time?
  - Detecting collinear triples in $\mathbb{R}^2$ in $o(n^2)$ time?
  - How to verify no instances?

Thank you for listening